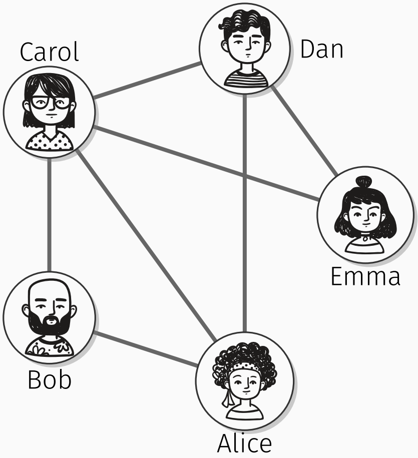
Descriptive Complexity of Learning

Steffen van Bergerem

June 27, 2024

Learning from Examples



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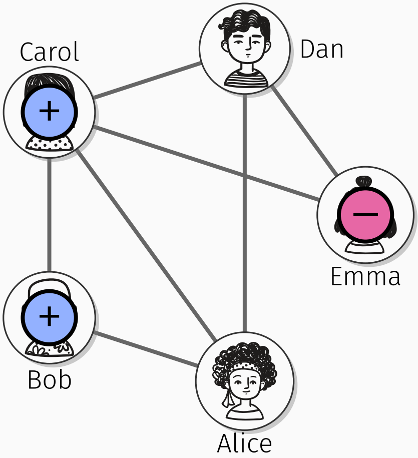


1

Positive examples

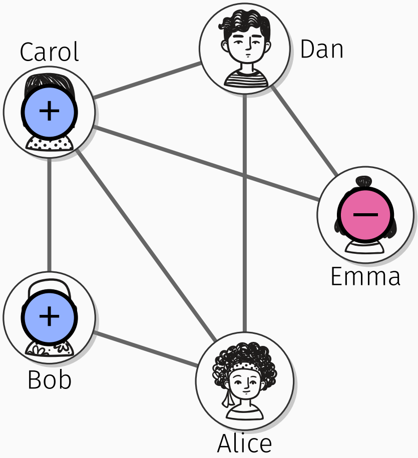
Bob

Carol

Negative examples

Emma

Learning from Examples



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Positive examples

Bob
Carol

Negative examples

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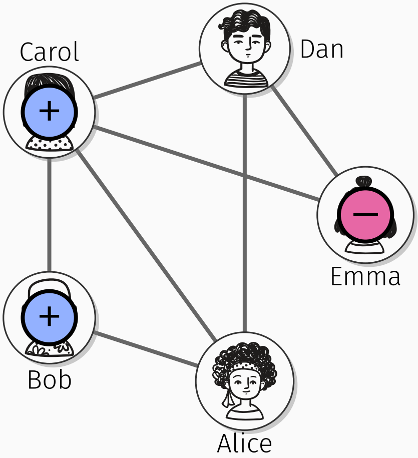
Possible hypothesis

Alice's friends

Learning from Examples



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Positive examples

Bob

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Negative examples

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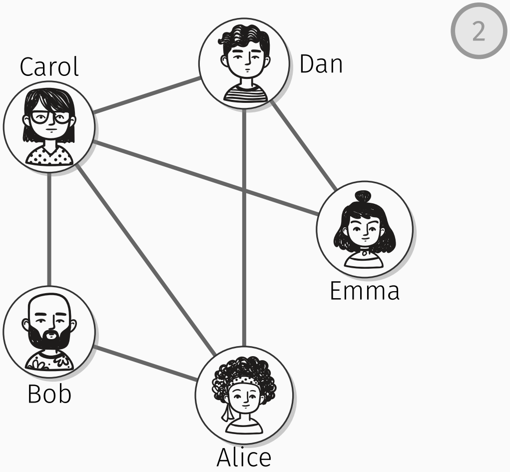
Possible hypothesis

Alice's friends

$\varphi(x) = E(x, Alice)$

Learning from Examples



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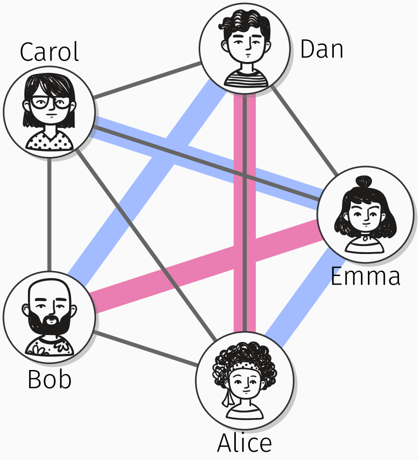


2

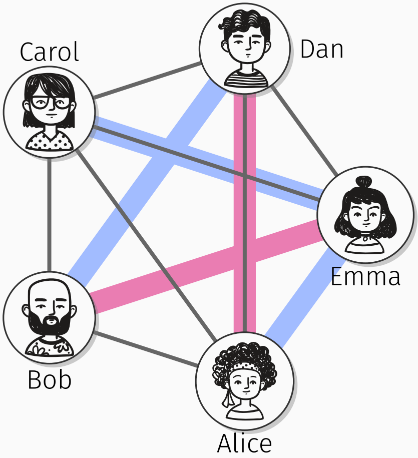
Positive examples

(Alice, Emma)
(Bob, Dan)
(Carol, Emma)

Negative examples

(Alice, Dan)
(Bob, Emma)

Learning from Examples



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Negative examples

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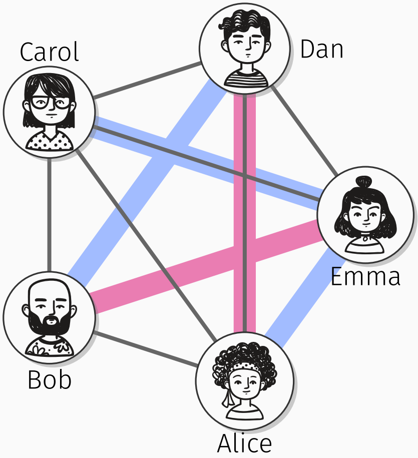
Possible hypothesis

having a common friend who is not Carol

Learning from Examples



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Negative examples

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Possible hypothesis

having a common friend who is not Carol

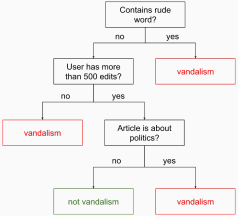
$\varphi(x_1, x_2) = \exists z \ (E(x_1, z) \land E(x_2, z) \land z \neq \text{Carol})$

Supervised Learning

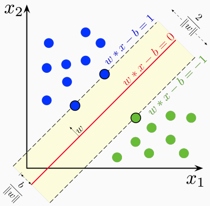
- learn from labelled examples

Supervised Learning

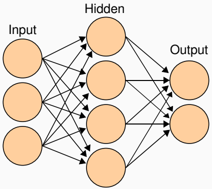
- learn from labelled examples
- algorithms with corresponding hypothesis specifications



decision trees



support-vector machines



neural networks

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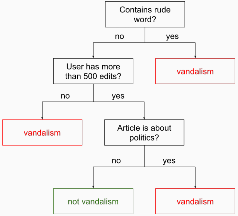

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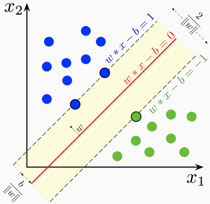

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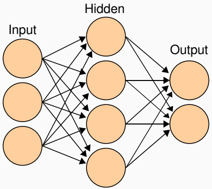


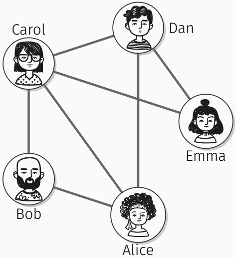
neural networks

- goal of this talk: complexity-theoretic analysis of the problem

→ **problem: specification of hypotheses**

Logical machine-learning framework

- introduced by Grohe and Turán (2002)
- inputs are labelled tuples from relational structure



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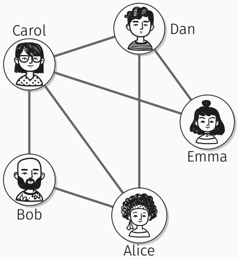

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- hypotheses are described using logics

$$\varphi(x_1, x_2) \quad = \exists z \left(E(x_1, z) \wedge E(x_2, z) \wedge z \neq \text{Carol} \right)$$

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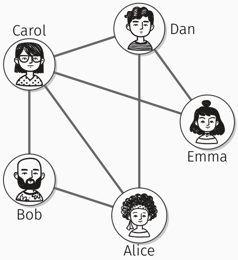

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- hypotheses are described using logics

$$\varphi(x_1, x_2; y) = \exists z \left(E(x_1, z) \land E(x_2, z) \land z \neq y \right)$$

parameter: Carol

Algorithmic Meta-Theorems

- **Grohe, Kreutzer, Siebertz (2014)**: properties definable in first-order logic (FO) can be decided in almost linear time on nowhere dense graph classes.

- Algorithmic meta-theorems yield efficient algorithms for certain kinds of logics on certain kinds of structures.

Algorithmic Meta-Theorems

- **Grohe, Kreutzer, Siebertz (2014)**: properties definable in first-order logic (FO) can be decided in almost linear time on nowhere dense graph classes.
- Algorithmic meta-theorems yield efficient algorithms for certain kinds of logics on certain kinds of structures.
- **Goal**: efficient algorithms for learning hypotheses described by certain kinds of logics on certain kinds of structures.

Learning First-Order Logic

Learning Problem

Learning Problem (for fixed $k, \ell, q \in \mathbb{N}$)

Input

- relational structure $\mathcal{A}$
- labelled examples $T = \big((\bar{v}_1, \lambda_1), \ldots, (\bar{v}_m, \lambda_m)\big)$

k-tuples

Output

- FO-formula $\varphi(x_1, \ldots, x_k; y_1, \ldots, y_\ell)$ with $\mathrm{qr}(\varphi) \leqslant q$
- parameters $w_1, \ldots, w_\ell \in U(\mathcal{A})$

with $\quad \mathrm{error}_T(\varphi, \bar{w}) \leqslant \min_{\varphi^*, \bar{w}^*} \mathrm{error}_T(\varphi^*, \bar{w}^*)$

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- $\varepsilon > 0$

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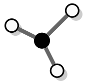
Statistical learning theory: Empirical Risk Minimisation Problem

Theorem (Grohe and Ritzert, 2017)

Concepts definable in FO on structures of small degree can be learned in sublinear time (in the size of the structure).



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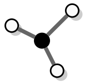

- FO in general <u>not</u> in sublinear time

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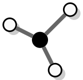

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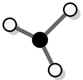

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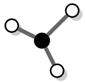

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2. more complex structures (v. B., Grohe, and Ritzert, PODS 2022)
   - parameterised complexity
   - nowhere dense structures

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Learning First-Order Logic

Theorem (Grohe and Ritzert, 2017)

Concepts definable in FO on structures of small degree can be learned in sublinear time.

We only consider formulas of limited quantifier rank.

```
for all formulas φ do
    for all parameters w̄ do
        compute error_T(φ, w̄)
return hypothesis φ, w̄ with minimum error
```

Theorem (Grohe and Ritzert, 2017)

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constantly many

1: **for all** normal forms of formulas φ **do**
2: **for all** parameters $\bar{w}$ **do**
3: compute $\mathrm{error}_T(\varphi, \bar{w})$
4: **return** hypothesis $\varphi, \bar{w}$ with minimum error

1) Gaifman normal form

Theorem (Grohe and Ritzert, 2017)

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We only consider formulas of limited quantifier rank.

constantly many

small degree,
small neighbourhood
(polylogarithmic)

1: **for all** normal forms of formulas φ **do**
2: **for all** parameters $\bar{w}$ in a certain neighbourhood **do**
3: compute $\mathrm{error}_T(\varphi, \bar{w})$
4: **return** hypothesis $\varphi, \bar{w}$ with minimum error

1) Gaifman normal form
2) Gaifman locality and Feferman-Vaught decompositions

Learning Concepts Definable in Logics

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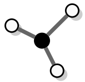

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   (v. B., Grohe, and Ritzert, PODS 2022)
   - parameterised complexity
   - nowhere dense structures

Learning FO with Counting

Concepts definable in FOCN
on structures of degree at most $(\log \log n)^{c}$
can be learned in sublinear time.

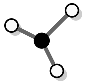
Theorem (v. B., 2023)

Concepts definable in FOCN on structures of degree at most $(\log \log n)^c$ can be learned in sublinear time.

- no Gaifman normal form for FOCN, only Hanf normal form
- number of formulas not constant any more

Theorem (Grohe and Ritzert, 2017)

Concepts definable in FO on structures of small degree can be learned in sublinear time (in the size of the structure).



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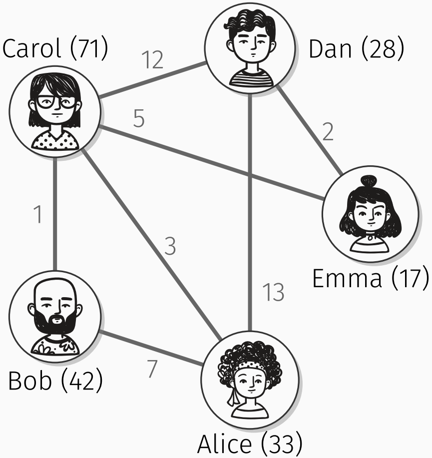
Learning FO with Weight Aggregation

Concepts definable in FOWA_1
on weighted structures of small degree
can be learned in sublinear time.

First-Order Logic with Weight Aggregation (FOWA)



Introduced in (v. B. and Schweikardt, CSL 2021)

Can build formulas such as

$$\varphi(x) = \Big(\sum_y a(y).E(x,y) \leqslant \sum_y a(y).\neg E(x,y)\Big)$$

Theorem (v. B. and Schweikardt, CSL 2021)

Concepts definable in $FOWA_1$ on weighted structures of small degree can be learned in sublinear time with quasilinear-time precomputation.

Theorem (v. B. and Schweikardt, CSL 2021)

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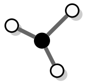

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- parameterised complexity
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Parameterised Complexity of Learning

In general, learning FO is hard.

But on nowhere dense classes,
learning FO is fixed-parameter tractable.

Parameterised Clique Problem

Input

- graph G
- $k \in \mathbb{N}$

Parameter

k

Output

"Yes" if and only if G has a k-clique

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Parameterised Complexity

- XP: running time $n^{f(k)}$
- fixed-parameter tractable (FPT): running time $f(k) \cdot p(n)$

Learning Problem (for fixed $k, \ell, q \in \mathbb{N}$)

Input

- relational structure $\mathcal{A}$
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- $\varepsilon > 0$

Output

- FO-formula $\varphi(x_1, \dots, x_k; y_1, \dots, y_\ell)$ with $\mathrm{qr}(\varphi) \leqslant q$
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- $\varepsilon > 0, \quad k, \ell, q \in \mathbb{N}$

Parameter

$k + \ell + q + 1/\varepsilon$

Output

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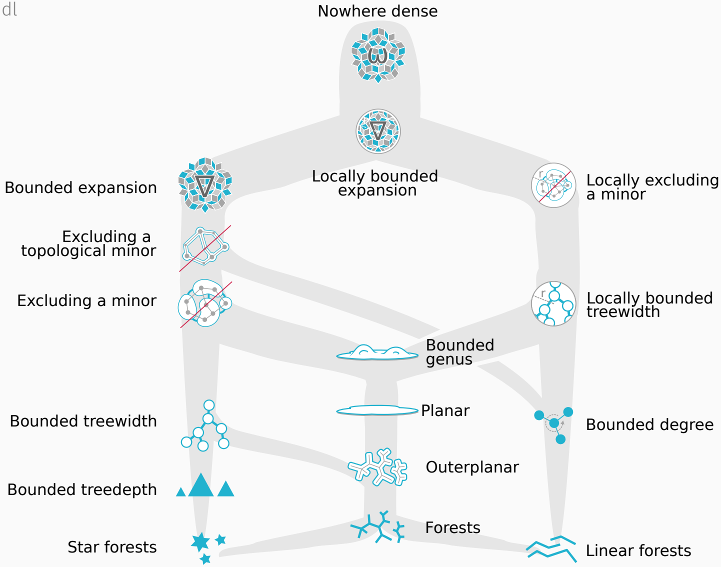
Theorem (v. B., Grohe, and Ritzert, PODS 2022)

Learning concepts definable in FO is at least as hard as the model-checking problem for FO.

Under a common complexity-theoretic assumption (p-Clique $\notin$ FPT), model checking is not fixed-parameter tractable.

Classes with a Tractable Model-Checking Problem

Illustration by Felix Reidl



Tractability of Learning

Theorem (v. B., Grohe, and Ritzert, PODS 2022)

For every nowhere dense class $\mathcal{C}$ of graphs, learning concepts definable in FO is fixed-parameter tractable on $\mathcal{C}$.

Proof uses game characterisation of nowhere dense classes.

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