HUMBOLDT-UNIVERSITÄT ZU BERLIN

On the VC Dimension of First-Order Logic with Counting and Weight Aggregation

Steffen van Bergerem and Nicole Schweikardt

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- measure for richness/complexity of set families

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- our main motivation: connections between VC dimension and learnability of concepts that are definable in logics
- in probably approximately correct learning (PAC learning): number of examples needed to learn a concept depends on the VC dimension
- Grohe and Turán (TOCS 2004) gave upper bounds for FO and MSO definable concepts on several classes of structures

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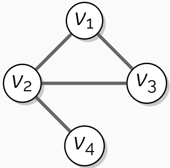
- VC dimension of $\mathcal{S}$ is maximum size of a set shattered by $\mathcal{S}$
- $\text{VCdim}(\mathcal{S}) = |\{1, 4\}| = 2$

VC Dimension of First-Order Logic

$$\varphi(\bar{x}, \bar{y}) = \varphi(x, y_1, y_2) = E(x, y_1) \ \vee \ x = y_2$$



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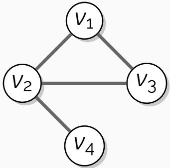

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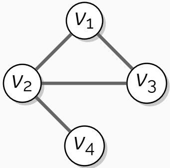

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v_1v_1	✓	✓	✓	✗
v_1v_2	✗	✓	✓	✗
v_1v_3	✗	✓	✓	✗
v_1v_4	✗	✓	✓	✓
v_2v_1	✓	✗	✓	✓
v_2v_2	✓	✓	✓	✓
v_2v_3	✓	✗	✓	✓
v_2v_4	✓	✗	✓	✓
v_3v_1	✓	✓	✗	✗
v_3v_2	✓	✓	✗	✗
v_3v_3	✓	✓	✓	✗
v_3v_4	✓	✓	✗	✓
v_4v_1	✓	✓	✗	✗
v_4v_2	✗	✓	✗	✗
v_4v_3	✗	✓	✓	✗
v_4v_4	✗	✓	✗	✓

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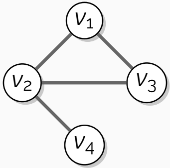

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$v_2 v_1$	✓	✗	✓	✓
$v_2 v_2$	✓	✓	✓	✓
$v_2 v_3$	✓	✗	✓	✓
$v_2 v_4$	✓	✗	✓	✓
$v_3 v_1$	✓	✓	✗	✗
$v_3 v_2$	✓	✓	✗	✗
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$v_3 v_4$	✓	✓	✗	✓
$v_4 v_1$	✓	✓	✗	✗
$v_4 v_2$	✗	✓	✗	✗
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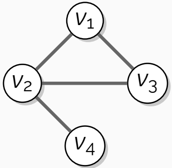

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- $\text{VCdim}(\varphi, G) \in \mathcal{O}(\log |V(G)|)$ for all φ and G

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VC Dimension of First-Order Logic

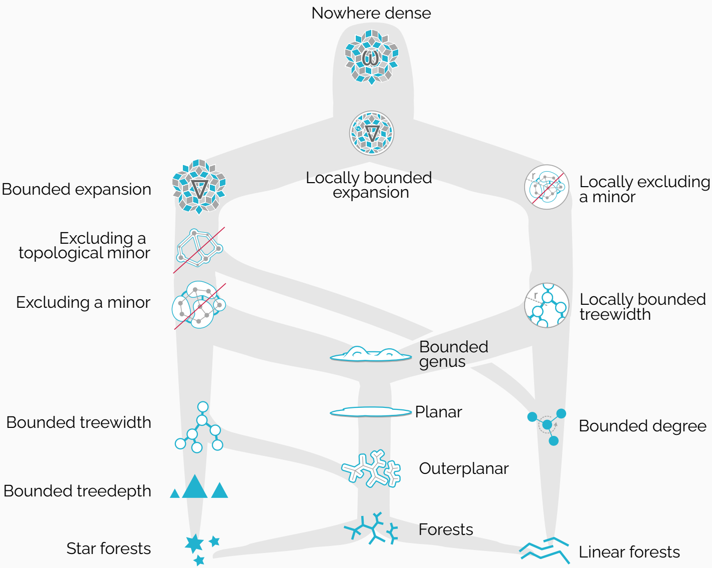
Let $\mathcal{C}$ be a nowhere dense graph class, and let $\varphi(\bar{x}, \bar{y})$ be an FO formula.

Adler and Adler, 2014

There is a constant $d \in \mathbb{N}$ such that $\mathrm{VCdim}(\varphi, G) \leqslant d$ for all $G \in \mathcal{C}$.

Classes of Sparse Graphs

Illustration by Felix Reidl



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The VC density of φ in G is at most $|\bar{x}|$ for all $G \in \mathcal{C}$. This bound is optimal.

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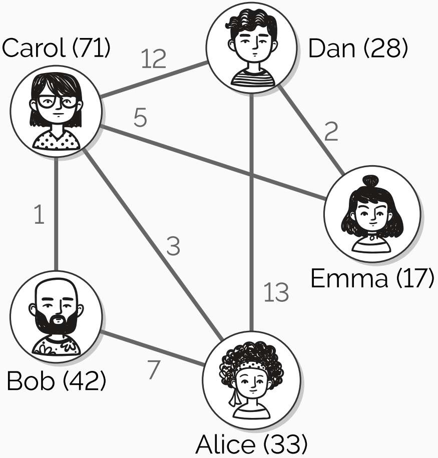
v. B. and Schweikardt, 2025

All of the above also hold for nowhere dense classes of vertex- and edge-**weighted graphs** and FOC_1 and $\mathbf{FOWA}_1$ **formulas**.

First-Order Logic with Weight Aggregation (FOWA)



Introduced in (v. B. and Schweikardt, CSL 2021)

Terms

- $t(x) = \sum_y a(y) \cdot \ell(x,y).E(x,y)$

Formulas

Icons made by Freepik from Flaticon

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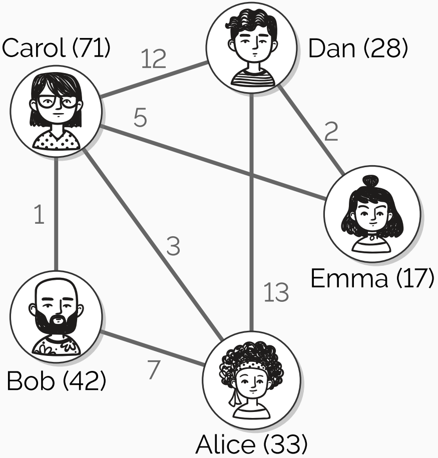

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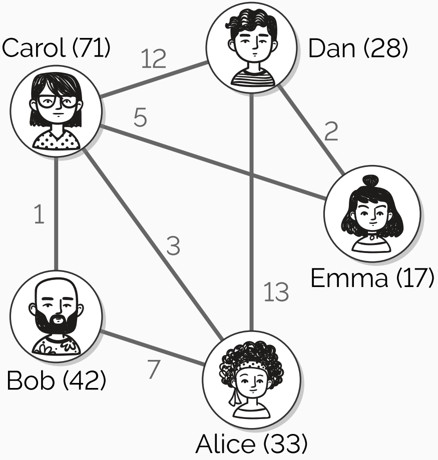

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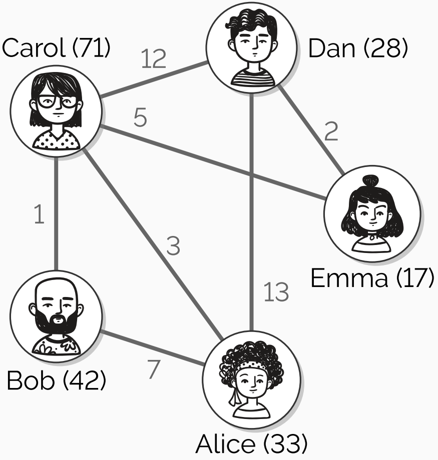

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FOWA_1

- subformulas comparing terms may only have at most one free variable

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Main results

Let $\mathcal{C}$ be a nowhere dense class of vertex- and edge-weighted graphs, and let $\varphi(\bar{x}, \bar{y})$ be an FOC_1 or FOWA_1 formula.

v. B. and Schweikardt, 2025

There is a constant $d \in \mathbb{N}$ such that the **VC dimension** and the **ladder index** of φ in G are at most d for all $G \in \mathcal{C}$.

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This lifts a similar result from (Pilipczuk, Siebertz, and Toruńczyk, 2018) from FO to more expressive logics on more expressive graph classes.

main tool: locality results for FOWA_1

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There is an FOC_2 formula with **unbounded VC dimension** on the class of all unranked trees of height at most 3.