Evaluation of First-Order Logic with Counting on Sparse Classes

HUMBOLDT-UNIVERSITÄT ZU BERLIN

Steffen van Bergerem

Finite and Algorithmic Model Theory 2025

First-Order Model Checking

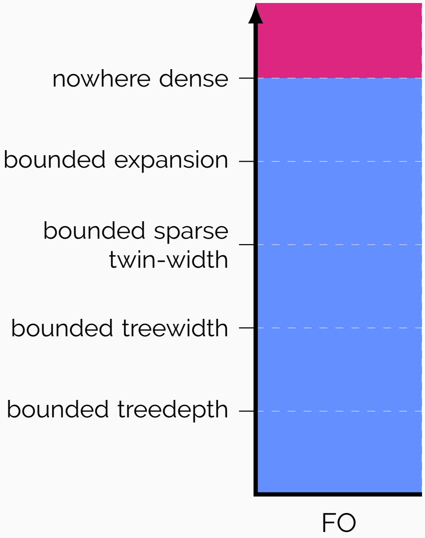
Given a graph G and an FO sentence φ

Decide whether $G \models \varphi$

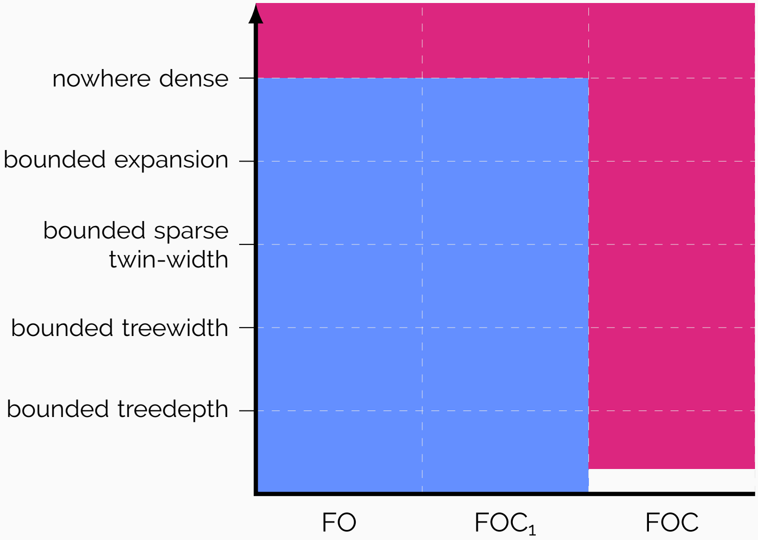
Given a relational structure $\mathcal{A}$ and an FO sentence φ

Decide whether $\mathcal{A} \models \varphi$

Model Checking on Sparse Classes



Model Checking on Sparse Classes



First-Order Logic with Counting (FOC)

Counting terms

- i for every integer $i \in \mathbb{Z}$
- $\#(y_1, \ldots, y_k).\varphi(\bar{x}, \bar{y})$ for every FOC formula $\varphi(\bar{x}, \bar{y})$
- $t_1 + t_2$ and $t_1 \cdot t_2$ for all FOC counting terms t_1, t_2

Formulas

First-Order Logic with Counting (FOC)

Counting terms

- i for every integer $i \in \mathbb{Z}$
- $\#(y_1, \ldots, y_k).\varphi(\bar{x}, \bar{y})$ for every FOC formula $\varphi(\bar{x}, \bar{y})$
- $t_1 + t_2$ and $t_1 \cdot t_2$ for all FOC counting terms t_1, t_2

Formulas

- every FO formula
- $\neg\varphi_1,\ \varphi_1 \wedge \varphi_2,\ \varphi_1 \vee \varphi_2,\ \exists x \varphi_1$ for all FOC formulas φ_1, φ_2
- $P(t_1, \ldots, t_m)$ for all FOC counting terms $t_1, \ldots, t_m$ and $P \in \mathbb{P}$, $P \subseteq \mathbb{Z}^m$

First-Order Logic with Counting (FOC)

Counting terms

- i for every integer $i \in \mathbb{Z}$
- $\#(y_1, \ldots, y_k).\varphi(\bar{x}, \bar{y})$ for every FOC formula $\varphi(\bar{x}, \bar{y})$
- $t_1 + t_2$ and $t_1 \cdot t_2$ for all FOC counting terms t_1, t_2

Formulas

- every FO formula
- $\neg\varphi_1,\ \varphi_1 \wedge \varphi_2,\ \varphi_1 \vee \varphi_2,\ \exists x \varphi_1$ for all FOC formulas φ_1, φ_2
- $P(t_1, \ldots, t_m)$ for all FOC counting terms $t_1, \ldots, t_m$ and $P \in \mathbb{P}$, $P \subseteq \mathbb{Z}^m$

FOC$_1$

- introduced by Grohe and Schweikardt (PODS 2018)
- last rule may only be applied if $|\bigcup_{i=1}^{m} \text{free}(t_i)| \leqslant 1$

First-Order Logic with Counting (FOC)

Counting terms

- $t_1(x) = \#(y).E(x,y)$
- $t_2 = \#(x_1,\ldots,x_k).(\bigwedge_{1\leqslant i<j\leqslant k} E(x_i,x_j))$

Formulas

First-Order Logic with Counting (FOC)

Counting terms

- $t_1(x) = \#(y).E(x, y)$
- $t_2 = \#(x_1, \ldots, x_k).(\bigwedge_{1 \leqslant i < j \leqslant k} E(x_i, x_j))$

Formulas

- $\varphi_1(x) = \big(\#(y).E(x, y) \leqslant \#(y).\neg E(x, y)\big)$
- $\varphi_2(x) = \exists y\, E(x, y) \wedge \big(t_1(x) \leqslant t_1(y)\big)$

First-Order Logic with Counting (FOC)

Counting terms

- $t_1(x) = \#(y).E(x,y)$
- $t_2 = \#(x_1, \dots, x_k).(\bigwedge_{1 \leqslant i < j \leqslant k} E(x_i, x_j))$
- $t_3 = t_2 + \#(x).\varphi_1(x)$

Formulas

- $\varphi_1(x) = (\#(y).E(x,y) \leqslant \#(y).\neg E(x,y))$
- $\varphi_2(x) = \exists y\, E(x,y) \wedge (t_1(x) \leqslant t_1(y))$

First-Order Logic with Counting (FOC)

Counting terms

- $t_1(x) = \#(y).E(x,y)$
- $t_2 = \#(x_1,\ldots,x_k).(\bigwedge_{1\leqslant i<j\leqslant k} E(x_i,x_j))$
- $t_3 = t_2 + \#(x).\varphi_1(x)$

Formulas

- $\varphi_1(x) = (\#(y).E(x,y) \leqslant \#(y).\neg E(x,y))$
- $\varphi_2(x) = \exists y\, E(x,y) \wedge (t_1(x) \leqslant t_1(y))$

FOC_1

- introduced by Grohe and Schweikardt (PODS 2018)
- subformulas comparing terms have at most one free variable

First-Order Logic with Counting (FOC)

Counting terms

- $t_1(x) = \#(y).E(x,y)$
- $t_2 = \#(x_1,\ldots,x_k).(\bigwedge_{1\leqslant i<j\leqslant k} E(x_i,x_j))$
- $t_3 = t_2 + \#(x).\varphi_1(x)$

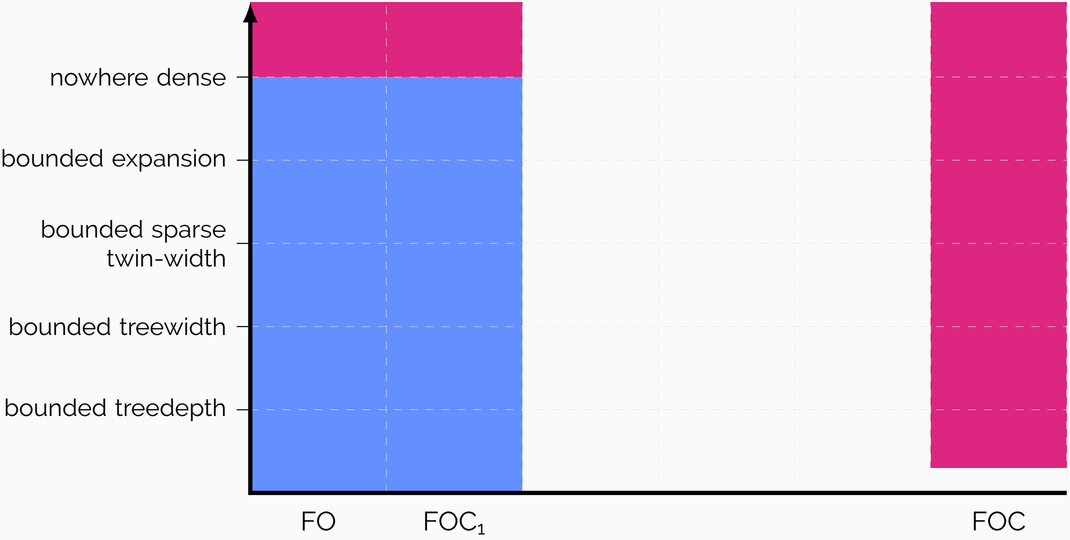
Formulas

- $\varphi_1(x) = (\#(y).E(x,y) \leqslant \#(y).\neg E(x,y))$
- $\varphi_2(x) = \exists y\, E(x,y) \wedge (t_1(x) \leqslant t_1(y))$

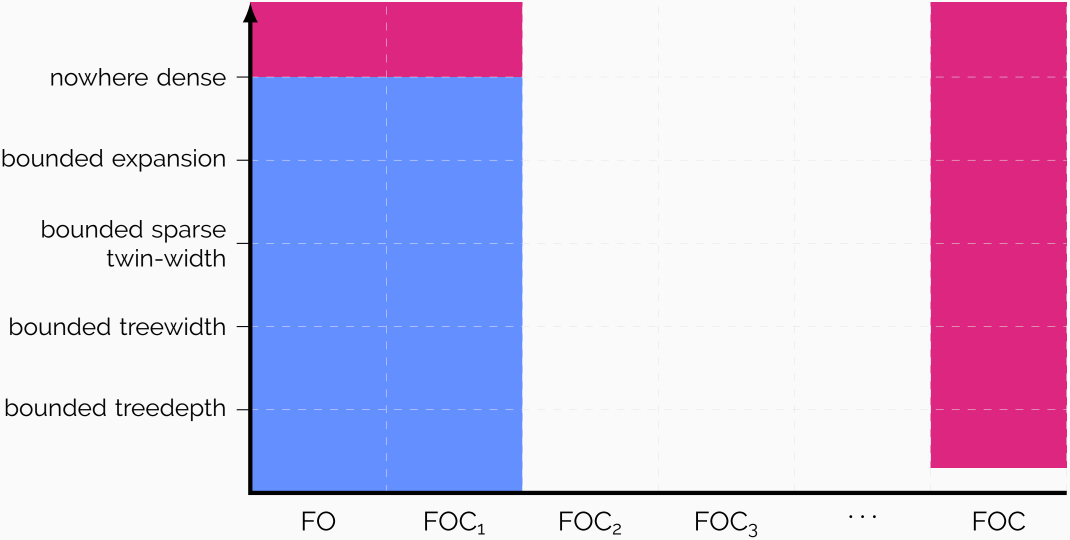
FOC_1

- introduced by Grohe and Schweikardt (PODS 2018)
- subformulas comparing terms have at most one free variable

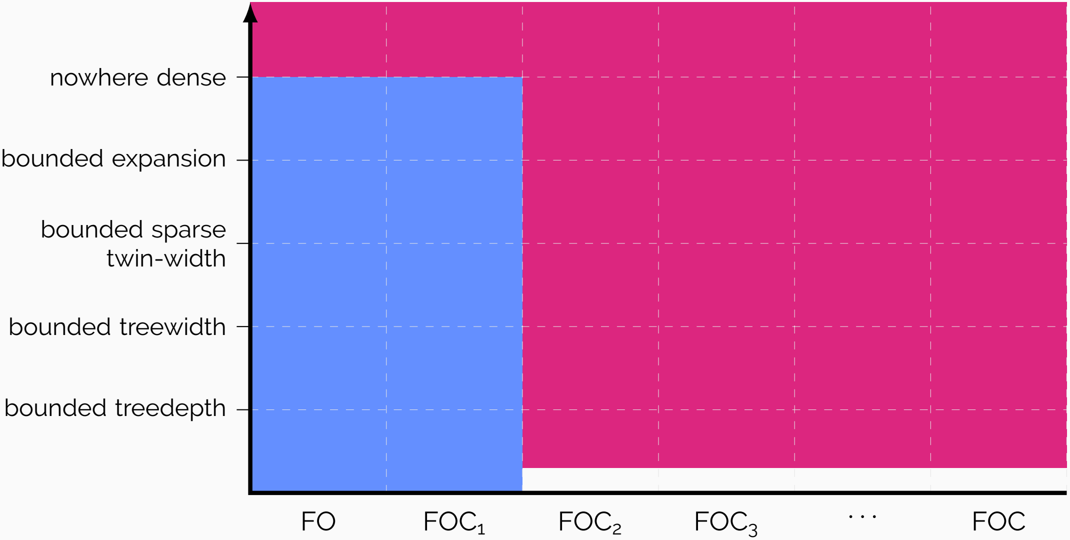
Evaluation on Sparse Classes



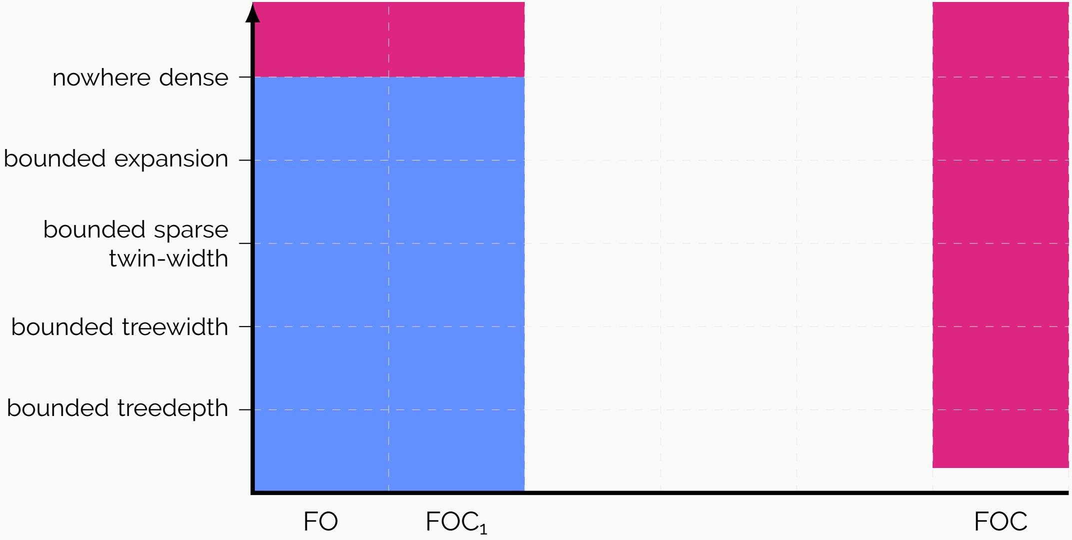
Evaluation on Sparse Classes



Evaluation on Sparse Classes



Evaluation on Sparse Classes



Guarded FOC

FOC_1
subformulas comparing terms have at most one free variable

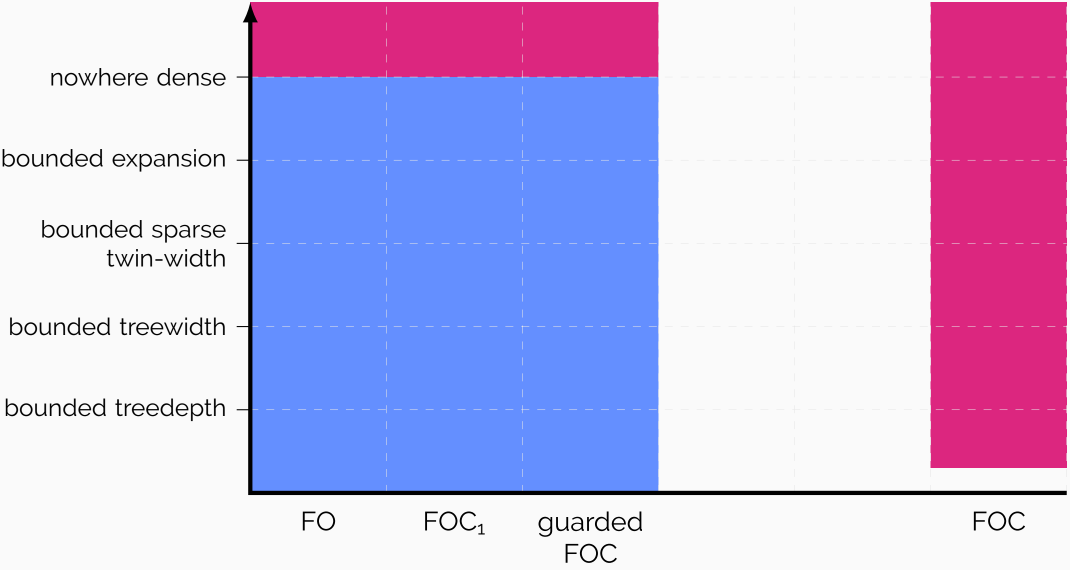
Guarded FOC

FOC_1
subformulas comparing terms have at most one free variable

guarded FOC
free variables in subformulas comparing terms are guarded by an atom

- $E(x, y) \wedge \big(t_1(x) \leqslant t_1(y)\big)$
- $R(\bar{x}) \wedge \big(t(\bar{y}) \leqslant t'(\bar{z})\big)$, where $\bar{y}, \bar{z} \subseteq \bar{x}$

Evaluation on Sparse Classes



Guarded FOC

FOC_1

subformulas comparing terms have at most one free variable

guarded FOC

free variables in subformulas comparing terms are guarded by an atom

- $E(x, y) \wedge \big(t_1(x) \leqslant t_1(y)\big)$
- $R(\bar{x}) \wedge \big(t(\bar{y}) \leqslant t'(\bar{z})\big)$, where $\bar{y}, \bar{z} \subseteq \bar{x}$

Guarded FOC

FOC_1
subformulas comparing terms have at most one free variable

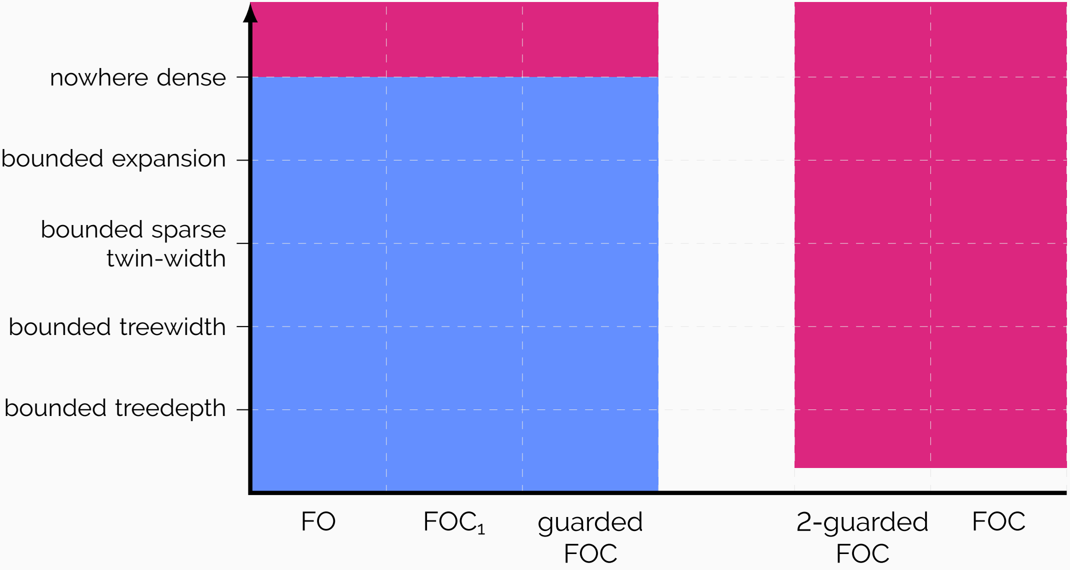
guarded FOC
free variables in subformulas comparing terms are guarded by an atom

- $E(x, y) \wedge (t_1(x) \leqslant t_1(y))$
- $R(\bar{x}) \wedge (t(\bar{y}) \leqslant t'(\bar{z}))$, where $\bar{y}, \bar{z} \subseteq \bar{x}$

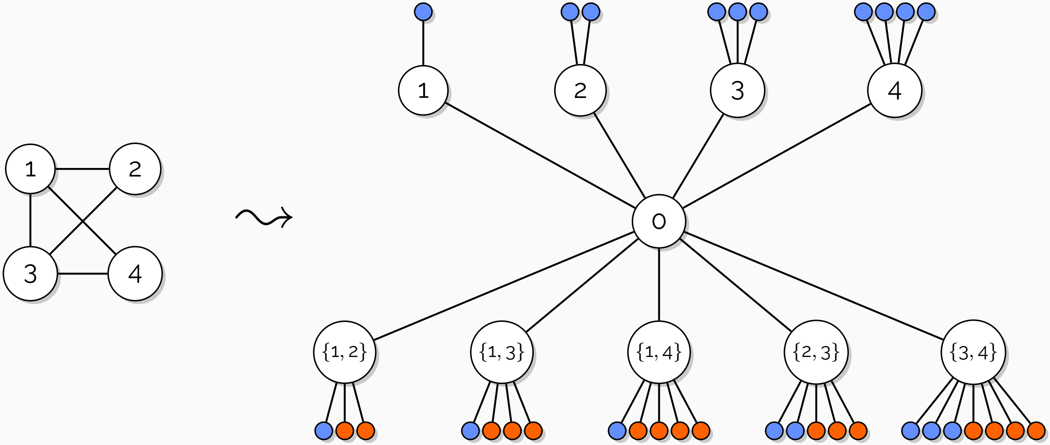
2-guarded FOC
free variables [...] are guarded by two intersecting atoms

- $E(x, y) \wedge E(y, z) \wedge (t_1(x) \leqslant t_1(z))$

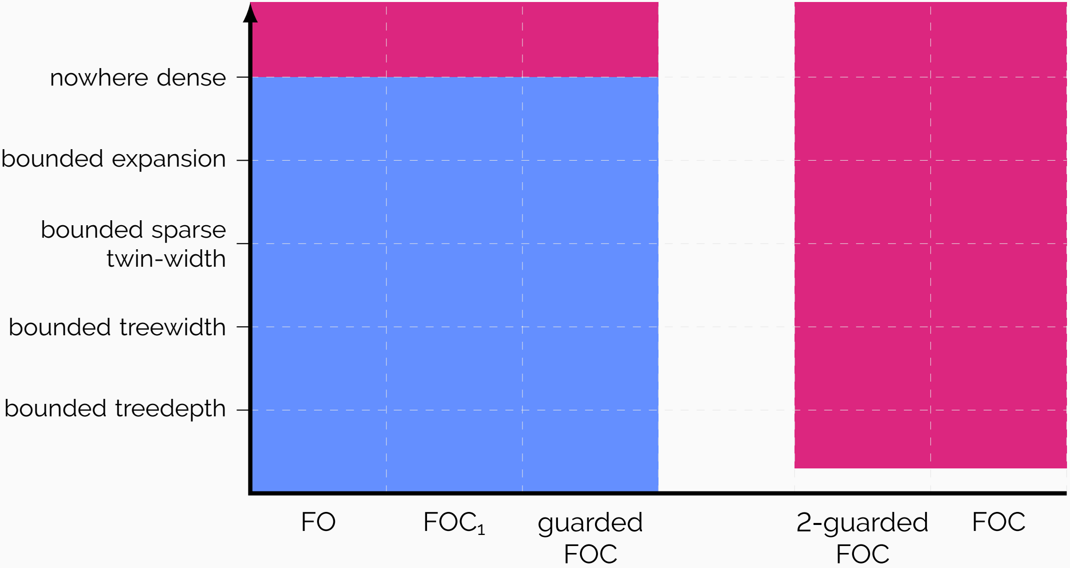
Evaluation on Sparse Classes



Hardness of 2-Guarded FOC



Evaluation on Sparse Classes



Guarded FOC

FOC_1
subformulas comparing terms have at most one free variable

guarded FOC
free variables in subformulas comparing terms are guarded by an atom

- $E(x, y) \wedge (t_1(x) \leqslant t_1(y))$
- $R(\bar{x}) \wedge (t(\bar{y}) \leqslant t'(\bar{z}))$, where $\bar{y}, \bar{z} \subseteq \bar{x}$

2-guarded FOC
free variables [...] are guarded by two intersecting atoms

- $E(x, y) \wedge E(y, z) \wedge (t_1(x) \leqslant t_1(z))$

Guarded FOC

FOC_1
subformulas comparing terms have at most one free variable

guarded FOC
free variables in subformulas comparing terms are guarded by an atom

- $E(x, y) \wedge (t_1(x) \leqslant t_1(y))$
- $R(\bar{x}) \wedge (t(\bar{y}) \leqslant t'(\bar{z}))$, where $\bar{y}, \bar{z} \subseteq \bar{x}$

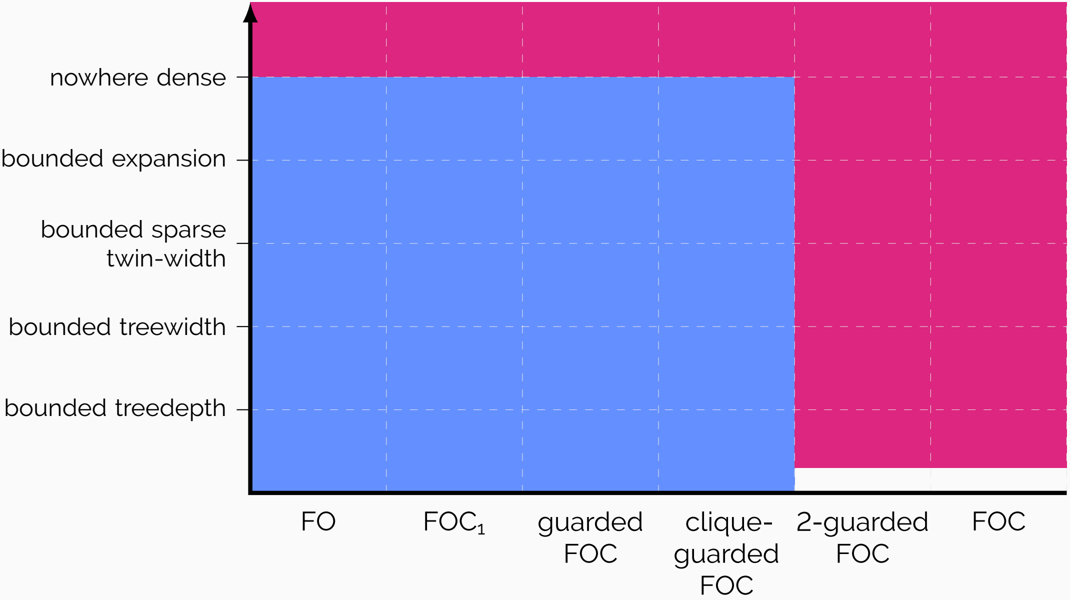
2-guarded FOC
free variables [...] are guarded by two intersecting atoms

- $E(x, y) \wedge E(y, z) \wedge (t_1(x) \leqslant t_1(z))$

clique-guarded FOC
every pair of free variables [...] is guarded by an atom

- $E(x, y) \wedge E(y, z) \wedge E(x, z) \wedge (t(x, y, z) \leqslant t'(x, z))$

Evaluation on Sparse Classes



Main Results

Tractabilty

For every effectively nowhere dense class $\mathcal{C}$ and every $\varepsilon > 0$,
there is an algorithm that solves model-checking and term-evaluation for clique-guarded FOC on every structure $\mathcal{A} \in \mathcal{C}$ in time $\mathcal{O}_{\mathcal{C},\xi,\sigma,\varepsilon}(|U(\mathcal{A})|^{1+\varepsilon})$.

Hardness

Model-checking for 2-guarded FOC is AW[$*$]-hard on the class of unranked trees of height at most 3.

Evaluation on Sparse Classes